Payback Time: Knowing it and Controlling it*

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Amortization is the process of payback, that is, retiring debt by borrowers. The counterpart, of course, is recovering capital by lenders. We are also interested here in the payback of equity capital ("equity" means the capital not borrowed, but supplied by the owner. It is also called "direct investment.") How fast do direct investors get back their money? "Capital recovery" is the term for that. **We will treat equity and debt together, and interchange the terminology**.

In cash flow management we are especially interested in how fast capital comes back to be reinvested. It is that flow of cash recovery that pays the bills, meets the payroll, keeps the wheels turning, and saves you from insolvency. Insolvency is an extreme form of illiquidity. It is not quite as bad as bankruptcy, but it can shut you down: it's when you run out of cash to meet current needs. Insolvency is when your short term liabilities > short term assets. (Bankruptcy is when your total liabilities > total assets.)

Capital in business is like the wheel of a bicycle: you have to keep it turning, or fall off. Macroeconomically, reinvestment in payments to workers generates most of the national income and makes jobs, so (to change the metaphor) it is important to the whole fleet as well as to your little putt-putt.

In naive usage, payback time means simply $\frac{P}{a}$, as though all cash flow from an investment could be used to retire principal. In your dreams! You hear that a lot, though, often from salesmen of major appliances. "You'll get all your money back in savings in the first 5 years; after that it's all clear profit." People selling energy-saving devices during the energy crunch (1978 or so) used it to death. Anything in a good cause, I suppose. I support conservation, too, but this was ridiculous. You should be able to see right through it. If you financed that new, energy-saving air conditioner on time, your bank would help you see through it soon enough. You'd have to pay interest on the loan before putting anything aside to repay it. Here we will develop correct ways to figure payback time.

Your bank lender is not being unreasonable. Suppose you put your money permanently in a bank savings account. Now, you are the creditor. You expect the bank to pay you interest forever. It does not occur to you to regard the interest payments as coming from your principal. Only withdrawals in excess of interest come from principal. Interest comes first. When the bank lends to you, it sees things the same way.

Payback time can be simple to figure on that savings account. If you deposit a lump of, say, $1,000, draw interest, and then withdraw it after 6 years, the payback time is exactly 6 years. That's also called the "period of investment." Redeemable bonds are like that, too (unless you buy them for more or less than their redemption value (R)). Some real assets fit that model pretty well, too, because they yield a flow of cash or service, and then have a high salvage value. Cows, sows, and hens are examples. The cow neither appreciates like a tree, nor depreciates like a building: it is like a simple savings account from which you withdraw the interest regularly. Other cases are not so simple, though.

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* WP 003

With PIPO ("point input-point output") investments we said nothing about amortization, because you recover your first point outlay all at once, at the "term" (i.e. the end, the time of "point-output"). There's more to it, though. Your principal went in at the start, but capitalized interest, an invisible input, goes in one year at a time. To figure the overall payback time, we would take an average value of *n* based on when each bit of interest was capitalized. In addition, imputed ground rent has to be capitalized each year. This is not as hard as you might think, but we defer it for now[1].

We deal here with debts being retired on the installment plan, i.e. with level payments over *n* years. The level payment, *a*, is determined using the IPF ("installment plan factor"), as by now you know so well. The principles we learn about paying debts also apply to real assets whose cash or service flows remain fairly constant over long periods: assets like buildings. It is a fair inference that buildings depreciate (fall in value) over their lives in a pattern similar to the fall in the balance of a fully amortized loan—otherwise bankers would not make such loans in the first place.

Before you can apply revenues to paying off debt, you must meet other priorities. When a business earns revenues, the creditors line up in a certain priority, both logically and legally. If you're renting, rent comes first—in fact, you usually pay in advance. They've gotcha. Without a location, you're out of business—that's basic.

Prior claims or liens are called "senior"; later ones, "junior." Junior claims are called "subordinate" to senior claims. If the debtor asks you to "subordinate" your claim, look out! He thinks you were born yesterday—he's putting you at greater risk. Banks are very touchy on this point. In fact, being large bureaucratic institutions, where rules are rules and loan officers are peons, they can be unreasonably touchy. Often, they will not subordinate their loans, even though the senior lien be only for $1.

As to taxes, it depends on which taxes, and it varies a lot with times and customs. Taxes generally come first, but politicians are malleable, and some creditors are better organized than others. Battles have been waged, and fortunes won and lost, over the priority of different taxes, and benefit assessments. I could tell you stories ... I will not try to summarize the complexities here. If you become a broker, treasurer, comptroller, lender, or borrower, this will always be more than a minor detail to you. For some lawyers and bankers, it is a career. For some savers, it has been a tragedy.

Current operating expenses have high priority—you have to keep the concern[2] going. Maintenance goes with operation, unless you get desperate for cash and are milking your old assets. The residue after that is called cash flow. In this course we generally finesse those prior steps, and begin with cash flow.

Cash flow goes first to pay interest, and only after that to retire the balance of debt. A business should treat its equity capital the same way, as though it were a debt to itself on which interest is

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[1]See the paper, "Normalized Models and Payback Time."

[2]"Concern" generally means a business, but can refer to any operation, e.g. a government, or other institution.

due. Otherwise, the owners are getting no return on their equity capital.[3] In these notes I will use the word "balance" to mean either of those.

Interest has first call on cash flow. The Riverside City Charter, for example, specifies in detail the priority of different kinds of obligations. It gives interest high seniority, to be sure the City can always borrow at favorable rates.[4] Interest is always based on the unpaid balance of a debt.

Likewise your creditor, in dealing with the I.R.S., has to report most of your payment as taxable interest, before assigning the residuum as payback of principal (which is not taxable). Creditors would love to report all payments as payback until the whole debt shall have been retired, so they can defer taxation as long as possible; but the I.R.S. won't let them. In this case, the I.R.S. is practicing good accounting, and forcing it onto others, willing or not. After all, if the debt were really all retired, there'd be no more reason to pay interest.

Likewise, for your equity investments, you owe yourself interest on their unrecovered balance.[5] Only after you have paid current interest (call it profit, if you prefer) can you apply any overage to reduce the balance, i.e. to say you've recovered your principal. In this case, though, I.R.S. practice is totally wrong and misleading, inconsistent with the foregoing paragraph, and muddies the waters. The I.R.S. lets owners deduct depreciation first, before declaring any taxable income, often creating an illusion that there is no income. This illusion results from political pressure to avoid taxes: there is no accounting reason for it. Don't get it mixed up with managerial accounting. You're not really recovering your capital as fast as you tell the I.R.S. it is depreciating.

Know these standard terms. For equity capital, payments or accounts that represent return of capital are called "capital consumption allowances" (CCAs), and also "depreciation" allowances. For debts, they are "payments on principal," or "redemptions." Payments that include both interest and redemption are called "debt service." Repaying debt with interest is "amortization." Loan repayment plans with level annual payments calculated to amortize a debt are called "fully amortized."

We deal here with debts being retired on the installment plan, i.e. with level payments over n years. The level payment, a, is determined using the IPF, as by now you know so well. The question now is, how do we divide that level payment, a, between interest and payback?

a. Interest vs. Recovery in year one

In year one the interest is obviously Pi. Solve the DCF equation for Pi, with I=1+i, and we have:

(1) $$Pi = a(1 - I^{-n})$$

To find interest as a share of the total payment, divide Pi by a:

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[3]I say they "should" treat equity this way. That does not mean they always do. Managers of large concerns are tempted to treat such equity as free, or low-cost. When they do, it is a sign of mismanagement. It is, indeed, one of the most common forms of mismanagement. Again, I could tell you stories ... and I may.

[4]This is a mixed blessing. The low rates are nice, of course, but the temptation is great to abuse this access to credit.

[5]If you try to skip an interest payment, and pay back principal instead, what happens? The unpaid interest is capitalized, i.e. added to the principal. Might as well pay interest first and avoid such mickey-mouse.

(2) $$\frac{Pi}{a} = 1 - I^{-n}$$

Repayment (*R*) as a share of the total is obviously the complement of *Pi*:

(3) $$\frac{R}{a} = I^{-n}$$

For example, on a 30 year note at 12%, I^{-n} = 1.12^{-30} = .033. That means only 3.3% of the first year's payment goes to reduce the balance owing. The rest is all interest.

You may forget how we began with Eqn. (2). You may think that in the first year you reduce the balance by 3.3%—that's WRONG! What I said, and what Eqns. (2) and (3) say, is that only 3.3% of the first year's payment goes to reduce the balance. To get the fractional drop of the balance, multiply I^{-n} by *a*, and divide it by *P*. (Explain why that gives you the fractional drop in the balance.)

(4) $$\frac{a}{P} = \frac{i}{1 - I^{-n}}$$

Multiply (4) by I^{-n}, and here is what you get:

(5) $$I^{-n} \cdot \frac{a}{p} = I^{-n} \cdot \frac{i}{1 - I^{-n}} = \frac{i}{I^{n} - 1}$$

Note, in passing, that the rightmost term is our old friend, the SFF, "sinking fund factor." These patterns have a way of repeating, and cropping up all over.

Now *that* is a small, small fraction indeed. Figure it out, with i=.12, and n=30. I get .004144. Do you? What percentage is that? Not much, is it? This is why you can pay for several years on a 30-year note, and lower the balance by no more than a tad. You might think you'd lower the balance in the first year by 1/30, since you've made 1/30 of your payments, but think again. You have only lowered it by 12.43% of 1/30 (.004044/.03333 = .1243). Yet that tad, small as it is, is what pays off the debt in 30 years, and saves you from perpetual debt. In each succeeding year the balance is a little lower, so the interest is a little less, leaving more to apply to the balance.

Perpetual debt is not a remote theory, unfortunately, but a common condition in human experience. Throughout much of history, people pledged their bodies (or their children's) for debt, and if they defaulted the creditor took the pledge as a slave. In early America, insolvent debtors were jailed including (occasionally) some rich and prominent leading citizens. After the Civil War and Reconstruction, slavery was illegalized, but "peonage" continued. A peon is a person in perpetual debt, lacking the modern escape route called bankruptcy. Peonage is still common in 3rd world countries. You've heard stories about loan sharks, and their collection techniques, which make good theatre but bad business.

We no longer pledge bodies for debt in this country, legally—or do we? The penalty for not paying personal income taxes is jail, and a high fraction of our taxes now go to pay interest on public debt. Something to think about, as we move ahead to see how fast we can retire debts.

b. Interest vs. Recovery in later years

(2 and 3) hold also for later years, but in place of the original *n* of 30 years you use the remaining term, *x* (for une*x*pired). $x = 30\text{-}c$, where *c* means *c*urrent year (*c* is the number of years elapsed). $x<n$; that makes R/a larger, as you move through time and $x \rightarrow 0$.

It is a comforting note that when c=29, x=1, and I^{-1} of your last payment goes to retire the balance of debt. At 12%, I^{-1} = 89.3%. (The other 10.7% is interest of 12% on the 89.3%—got that?) You do finally get out from under.

In place of the original *P*, use the current unpaid Balance, B_c.

To find B_c, it has to be the DCF ("discounted cash flow") of the remaining stream of payments. Use the DCFF ("DCF factor"), with *x* where originally you used *n*, or 30.

c. Balance remaining unpaid after half the total life

The remaining balance at *n*/2 is the present value of the level payment, *a*, over *n*/2 years. (Surprised? Remember that the original *P* is the present value of the level payment over *n* years, too; and if you think about it, this must always be true.)

(6) $$B_{\frac{n}{2}} = a\frac{1-I^{-\frac{n}{2}}}{i}$$

The unpaid balance as a share of *P* is:

(7) $$\frac{B_{\frac{n}{2}}}{P_0} = \frac{1-I^{-\frac{n}{2}}}{1-I^{-n}}$$

(7) is one of those inscrutable, unfamiliar functions it is helpful to tabulate, to get a feel for how it behaves. Table 1 shows the complement of (7), meaning [1-(7)], i.e., the share that *has* been repaid after half the life, for various original values of *n* (*n* is the original whole life of the debt or investment).

Table 1: Share repaid after half the full life, for various full lives (*n*), and various interest rates, *i*.

*i**n*	2	4	10	20	30	60	80
.04	.49	.48	.45	.40	.36	.24	.17
.08	.48	.46	.40	.32	.24	.09	.04
.12	.47	.44	.36	.24	.15	.03	.01
.16	.46	.43	.32	.18	.10	.01	.003

d. "Half-life": Time required to pay half the debt

This is different from half the time, which is *n*/2. As you have seen, debt is repaid slowly at first, then ever faster to the end. To visualize this distinction, draw a curve that slopes negatively, but that is nearly flat at first. It slopes more and more, and finally reaches a value of zero while sloping at an angle of roughly 135°. Label the graph, the Unpaid Balance of Debt Amortized in Level Installments. Label the curve B_c.

At $n/2$ on the abscissa draw a vertical line. Where it intersects the curve of B_c is the unpaid balance when $c=n/2$.

Now go up the ordinate from zero halfway to the y-intercept of B_c. From here draw a horizontal straight line. Where it intersects B_c is the half-life, the year when half the debt is repaid, or half the capital recovered.

Now you have a clear picture of what we are doing, let's do it with algebra, which will then yield us numbers: nice, comforting, "real" numbers. For this we must learn to solve DCFF for *n*. This requires logs, but don't panic, whip out your trusty modern hand calculator and learn what wonders you can work with this modest investment.

Starting with (2), here is how you solve for *n*.

$$\frac{Pi}{a} = 1 - I^{-n}$$

$$I^{-n} = 1 - \frac{Pi}{a}$$

Take the Ln of both sides:

$$-nLnI = Ln(1 - \frac{Pi}{a})$$

(8) $$n = \frac{-Ln(1 - \frac{Pi}{a})}{LnI}$$

"How can n be negative!?" I hear you scream. Not to worry:

$0 < Pi/a < 1$, so $[1-Pi/a]$ is positive and less than one. The log of a positive number less than one is negative. Everything is legal.

(8) must be easy to remember, because my students always do somehow. What I do is just remember the DCFF, and derive everything else from that. You do whatever works for you.

Patience, now, (8) does not give us the half-life we are seeking. I know you want to jump the gun and get this over with, but (8) is just step one. Gently, gently ... we'll get there, step by step.

Half the debt is retired (aka amortized, repaid, recovered, worked off, worked down, etc.) when P is reduced to half. So just replace *P* in (8) by *P*/2, and solve the equation for *n*. But this *n* is not the original *n*, so we will call it "*x*." Note that *x* is not half of *n*, it is the number of years of life remaining when half the debt is retired.

Warning! "*x*" is still not the final answer. We are looking for the half-life, which is the number of years elapsed. *x* is not that, but rather the life *remaining*. The half-life is n-x.

Some of you may forget that. I am already laying traps to catch you on the next exam. I am warning you all to watch it. You'll be distracted by what we do next, so when we finish, I'll remind you to look back here to where we started, and remember why we're doing this.

First, let's find *x*.

(9) $$x=\frac{-Ln(1-\frac{Pi}{2a})}{LnI}$$

Now, what is *x*, again? Doesn't help to find it if you forget what it is. If you've forgotten, check back and refresh. For an alternate way of finding *x*, see Appendix 1.

As I warned you, you may have been distracted by all that razzmatazz involved in finding *x*, and forget the last step. *You*, however, are going to remember. Let *h* = the half-life, then:

(10) $$h=n-x$$

Now, at last, we have done what we set out to do. And what was that? In all the technicalities, it's so easy to forget! Check back to be sure you know. Then, to check yourself, see note.[6]

It is interesting to see how the half-life compares with *n*, the whole life. Remember, half-life is not *n*/2, it is the year at which you repay half the debt. To get the half-life as a share of the whole life, divide by *h* by *n*:

(11) $$\frac{h}{n}=\frac{n-x}{n}=1-\frac{x}{n}$$

We went through this hassle to get numbers, so let's get some. They are in Table 2.

Table 2: Half-life/whole-life, for various *n* and *i*.

*i**n*	2	4	10	20	30	60	80
.04	.51	.52	.55	.60	.64	.74	.79
.08	.52	.54	.59	.68	.74	.85	.89
.10	.52	.55	.61	.71	.78	.88	.91
.12	.53	.56	.63	.74	.81	.90	.92
.16	.54	.57	.67	.78	.85	.92	.94

Table 2 tells you that if you buy on time at 12%, over 30 years, it will take 81% of 30 years to retire half your debt. That's 24.3 years, instead of 15 years.

Before organizing an indignant debtors' protest march, look at it from the lender's view. You are sitting on his money. He can't make any new loans with it until you return it. He can't generally mint more (in spite of what you have sometimes heard about bankers). If he wants more, he has to pay interest for it too, and/or render services.

Maybe you won't cry much for him, but look at it from the viewpoint of other borrowers: you are crowding them out. The lender can't lend them his capital until you return it. That's the social role of charging interest, to hurry you along so you won't sit on capital forever. The USSR tried running an economy without charging interest, and look what happened to them. Last time I gave

[6]Our objective is to find the half-life, *h*, corresponding to a whole-life, *n*, of a debt that is fully amortized on the installment plan. The half-life is our term for the age when the balance is reduced by 1/2.

a talk in Russia, my excellent translator fielded every technical term and curve ball I threw, until she surprised me by tugging at my sleeve and asking, “What iss thiss ‘compound interest’?” It is a new concept for them. Interest is the cost of time, that’s what it is. Without it, time is free, and means nothing.

Look at it from the equity investor’s viewpoint, when he sinks his capital into durables that resemble a 30-year mortgage in their payout schedules. He can’t reinvest his capital until he recovers it. His volume depends on capital turnover. This is a good reason to avoid building your version of The Trump Tower until you are very, very solvent.

They used to say there are lots of children running around without shoes on because their daddies forgot to lead out trump; but there are more today driving with bald tires because their daddies forgot to schedule the cash flow. “The Donald” may become one of those, along with a number of overextended local venturers whose names you may read in the press from time to time as their creditors close in. (Update, 1996: “The Donald” seems to have escaped. Some others did not, but I have made enough enemies this week, and will not name them here.)

Back to business, look at it from the employee’s viewpoint. Payrolls vary with volume. You’re sitting on his job. When God created Heaven and Earth he created compound interest, to keep capital busy turning and people in jobs.

Most income is normally consumed. The bulk of future investment has to come from *R*: most of this is normally recycled. The reason the people who have most of the capital still have it is because they habitually avoid impairing its substance, even when it passes through their hands as cash. Prodigal folks will, of course, sometimes “eat the seed corn,” but, having done so, they are out of it as capitalists. The ones that remain relevant are those that Recover and Recycle it. May you, Mr. and Ms. Future Manager, be one of those.

e. Share of interest in cash flow.

It seems to follow from all the above that a large fraction of cash flow (a) is net income to the investor, over and above recovery of capital or principal (P). How much is recovery of P?[7]

In the reading on “DCF,” section on Finite Flows, I had you draw a “cume curve” showing the growth of DCFF with n. This curve heads up bravely at first from the origin, at nearly 45°, but then bends over and flattens out, approaching a horizontal asymptote. Label it “P”. Draw another curve, a straight line heading up at 45° from the origin and never bending. This curve shows n, and is the cumulative cash flow. Assume $a = 1$ and label the curve “n”. A point on the P-curve tells you what P is normally required to buy an income stream whose total value is n. All the cash flow above P has to be interest. The vertical gap between curves widens as n rises. The curves thus make it clear that the share of interest in cash flow is an increasing function—a rapidly increasing function—of n.

The curves give a conspectus. Curves are most useful when used in double harness with the curves’ algebraic definitions, which in turn generate real numbers (the numbers are no more real than the algebra, but may seem that way because you are more used to them). When you have graphs, equations and numerical tables reinforcing each other you really get a handle on things.

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[7] *P* stands not just for Principal, but also for Present Value, Purchase Price, Planting Cost, Primary Outlay, etc. -- handy that they all start with P.

The cash flow above P is n-P. The share of interest in the total cash flow is therefore:

(12) $$\text{Share of interest} = \frac{n-P}{n} = 1 - \frac{P}{n} = 1 - \frac{1-I^{-n}}{ni}$$

Table 3 gives the numbers from (12).

Table 3: Share of interest in cash flow

$i\backslash n$	5	10	20	40	80
.05	.13	.23	.38	.57	.76
.10	.24	.39	.57	.76	.88
.15	.33	.50	.69	.83	.92
.20	.40	.58	.76	.88	.94

Table 3 says that if you borrow at 10%, amortized over 40 years, then 76% of all the payments you make go to pay interest. The other 24% pays off the loan. It's a hard fact of life: deal with it.

On the other hand, if you can pay up front, there is good news. If you buy a house for $100,000 and it yields you a service flow worth $10,000/year (in foregone rent), then in 40 years you will have received $400,000 from it, for your outlay of just $100,000. (The even better news is that the service flow is not taxable, while the interest on your loan is deductible. But wait, there is more bad news: that is why houses cost so much to buy.)

An operation is "capital-intensive" when a high share of cash flow goes to feed the banker or investor. That occurs, basically, when payments are long deferred, strung out in the far future. Normally that occurs when capital is durable, heavy and expensive, like the Rolls-Royce in Reading III,E,3.

f. Summary on payback

Here is a summary of what we have done with Eqns. (2)-(12), w.r.t. a 30-year loan at 12%.

From the IPFF, the level annual payment is at 12.41% of the Principal, P. Thus it is just a sliver above the $\underline{i}$ of 12%. But if that sliver (last time I called it a "tad") were lowered to zero, you'd be in debt forever.

From (3), the share of the first year's pmt that is not interest is only 3.3%. The symbol for this portion is R, which you may read as Redemption, Retirement, Recovery, Repayment, Recoupment, etc., for they all mean about the same thing. The rest is all interest, a pure flow of expense to the debtor, and to the lender, income.

Interest is taxable to the payee, and deductible by the payor. Repayment of P is neither. You have to separate them on your Form 1040. Our math shows you how.

From (7), the share repaid in 15 years is 15% of P. The other 85% remains unpaid, a stone around the neck of the debtor, and to the lender, frozen capital which he cannot yet recycle.

In (8) and (9) we learned how easy it is to solve the DCFF for its exponent, n, using logs.

From (10 and 11), the half-life is 24.2 years, or 81% of the whole life.

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APPENDIX 1: Greater Accuracy in Defining “Half-life”

Students with fine-tuned math instincts will sense that half-life as defined here is not identical with the true mean life of the capital, defined scrupulously. Gaffney’s “half-life” is a bit unsophisticated and naive, they may sniff. Let them sneer: I plead guilty. My motive, however, is economic: the extra trouble of 100% accuracy is not worth the cost in time.

For greater accuracy, see the paper, “Calculating Mean Periods of Investment, Using Normalized Models.” The half-life as used above is a shade longer than the true mean period of investment, which in the above case is 22.2 years, or 74% of the whole life. The true mean period, P, comes from (16), below, which is presented for completeness, but not otherwise discussed here. (16) is derived, discussed and illustrated in my notes of 11/84, as revised 1/92, “Calculating Mean Periods of Investment, Using Normalized Models,” pp. 14-22, Eqns. (7-10), Tables 1,2.

(16) $$p = \frac{m}{1 - e^{-\rho m}} - \frac{1}{\rho}$$

where e is the base of natural logarithms and $\rho = Ln(1+i)$.

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APPENDIX 2

Alternative form of Eqn. (9), to find x.

You don’t need to know P or a to find x from (9). From (2), $\frac{Pi}{a} = 1 - I^{-n}$. Therefore:

$$1 - \frac{Pi}{2a} = 1 - \frac{1 - I^{-n}}{2} = \frac{2 - 1 + I^{-n}}{2} = \frac{1 + I^{-n}}{2}$$

Substitute that in (9), and:

(9,alt) $$x = \frac{-Ln(\frac{1 + I^{-n}}{2})}{LnI} = \frac{Ln2 - Ln(1 + I^{-n})}{LnI}$$